

On the topology of stationary black hole event horizons in higher dimensions

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ABSTRACT: In four dimensions the topology of the event horizon of an asymptotically flat stationary black hole is uniquely determined to be the two-sphere S^2 . We consider the topology of event horizons in higher dimensions. First, we reconsider Hawking's theorem and show that the integrated Ricci scalar curvature with respect to the induced metric on the event horizon is positive also in higher dimensions. Using this and Thurston's geometric types classification of three-manifolds, we find that the only possible geometric types of event horizons in five dimensions are S^3 and $S^2 \times S^1$. In six dimensions we use the requirement that the horizon is cobordant to a four-sphere (topological censorship), Friedman's classification of topological four-manifolds and Donaldson's results on smooth four-manifolds, and show that simply connected event horizons are homeomorphic to S^4 or $S^2 \times S^2$. We show that the non-simply connected event horizons $S^3 \times S^1$ and $S^2 \times \Sigma_g$ and event horizons with finite non-abelian first homotopy group whose universal cover is S^4 , are possible. Finally, we discuss the classification in dimensions higher than six.

KEYWORDS: Black Holes in String Theory, Supergravity Models, Black Holes.

Contents

1. Introduction	1
2. Hawking's theorem revisited	2
3. The topology of event horizons in higher dimensions	4
3.1 Five-dimensional black holes	5
3.2 Six-dimensional black holes	6
3.3 Comments on higher dimensions	9

1. Introduction

In four dimensions the topology of the event horizon of an asymptotically flat stationary black hole is uniquely determined to be the two-sphere S^2 [1].¹ Hawking's theorem [1] requires the integrated Ricci scalar curvature with respect to the induced metric on the event horizon to be positive. This condition applied to two-dimensional manifolds determines uniquely the topology.

Another way to determine the topology of the event horizon is via the so called topological censorship [2]. Mathematically it requires the horizon to be cobordant to a sphere via a simply connected oriented cobordism. For a two-dimensional horizon it means that there is a simply connected three-dimensional oriented manifold whose boundary is the oriented disjoint union of the horizon and the two-sphere. Topological censorship implies that the topology of the event horizon is that of the two-sphere S^2 [3].

The classification of the topology of the event horizons in higher dimensions is more complicated [4]. For instance, for five-dimensional asymptotically flat stationary black holes, in addition to the known S^3 topology of event horizons, stationary black hole solutions with event horizons of $S^2 \times S^1$ topology (Black Rings) have been constructed [5].

In this letter we will consider the topology of event horizons in dimensions higher than four. First, we reconsider Hawking's theorem [1] and show it continues to hold in higher dimensions. Using this and Thurston's geometric types classification of three-manifolds [6, 7], we find that if the induced metric on the horizon is locally homogeneous, the only possible geometric types of event horizons in five dimensions are S^3 and $S^2 \times S^1$ (for a related discussion see [8]). These geometric types may give rise to horizon topologies which are quotients of them by discrete subgroups of the isometry group.

In six dimensions we use the requirement that the horizon is cobordant to a four-sphere, Friedman's classification of topological four-manifolds and Donaldson's results on smooth

¹We consider the connected part of the event horizons.

four-manifolds, and show that simply connected event horizons are homeomorphic to S^4 or $S^2 \times S^2$. We find allowed non-simply connected event horizons $S^3 \times S^1$ and $S^2 \times \Sigma_g$ (Σ_g is a genus g Riemann surface), and event horizons with finite non-abelian first homotopy group, whose universal cover is S^4 . Finally, we will discuss the classification in dimensions higher than six.

The letter is organized as follows. In section 2 we will reconsider the positivity theorem of [1]. Following the steps of the proof in four dimensions we will find that it holds also for asymptotically flat stationary black holes in dimensions higher than four. In section 3, we will consider the condition that the integrated Ricci scalar curvature with respect to the metric induced on the event horizon is positive and the requirement that the horizon is cobordant to a sphere, and study their implications for the possible topologies of the event horizons in dimensions higher than four.

2. Hawking's theorem revisited

In this section we will reconsider Hawking's theorem determining the two-sphere topology of the event horizon of four-dimensional asymptotically flat stationary black holes [1]. Following the steps of the proof in [1], we will find that also for asymptotically flat stationary black holes in dimensions higher than four the integrated Ricci scalar curvature $\hat{\mathcal{R}}$ with respect to the induced metric \hat{h} on the event horizon M_H , is positive

$$\int_{M_H} \hat{\mathcal{R}} d\hat{S} > 0. \tag{2.1}$$

The idea in the proof is to use the fact that the shear and divergence are zero at the horizon, but the divergence is positive outside the horizon. In the next section we will study the implications of (2.1) for the topology of the event horizon for higher-dimensional asymptotically flat stationary black holes.

Consider a stationary n -dimensional space-time \mathcal{M} with a metric g . Stationary means that there exists a one-parameter group of isometries whose orbits are time-like curves. \mathcal{M} is required to be regular predictable, i.e. its future is predictable from a Cauchy surface [1].

Denote by (Y_1, Y_2) two future-directed null vectors orthogonal to M_H , normalized as

$$Y_1^a Y_{2a} = -1. \tag{2.2}$$

We take Y_1 to be the future-directed null vector pointing out of the horizon, and Y_2 to be the vector pointing into the horizon. This still leaves us with the freedom to rescale

$$Y_1 \rightarrow e^y Y_1, Y_2 \rightarrow e^{-y} Y_2. \tag{2.3}$$

The (positive definite) induced metric on the horizon reads

$$\hat{h}_{ab} = g_{ab} + Y_{1a} Y_{2b} + Y_{2a} Y_{1b}. \tag{2.4}$$

We now deform the event horizon by moving each point on it a parameter distance ω along an orthogonal null geodesic with tangent vector Y_2^a . Following the same steps as in [1] one derives the equation

$$\frac{d\hat{\theta}}{d\omega} = p_{b;d}\hat{h}^{bd} - \mathcal{R}_{ac}Y_1^aY_2^c + \mathcal{R}_{adcb}Y_1^dY_2^cY_2^aY_1^b + p_a p^a - Y_{1;c}\hat{h}^c_d Y_2^d \hat{h}^b_a, \quad (2.5)$$

where $\hat{\theta} \equiv Y_{1;b}\hat{h}^b_a$ and $p^a = -\hat{h}^{ab}Y_{2c;b}Y_1^c$. Note that the last term on the RHS is zero on the horizon, as the shear and the divergence of the null geodesics with tangent vector Y_1 are zero there.

Using (2.3), $p^a \rightarrow p'^a = p^a + \hat{h}^{ab}y_{;b}$ and we get

$$\left. \frac{d\hat{\theta}'}{dw'} \right|_{w=0} = p_{b;d}\hat{h}^{bd} + y_{;bd}\hat{h}^{bd} - \mathcal{R}_{ac}Y_1^aY_2^c + \mathcal{R}_{adcb}Y_1^dY_2^cY_2^aY_1^b + p'_a p'^a. \quad (2.6)$$

Since $y_{;bd}\hat{h}^{bd}$ is the laplacian of y ($*d*dy$) on the $(n-2)$ -dimensional horizon, we can use a theorem of Hodge to set the first four terms on the RHS equal to a constant by a particular choice of y , as follows: The other three terms ($p_{b;d}\hat{h}^{bd} - \mathcal{R}_{ac}Y_1^aY_2^c + \mathcal{R}_{adcb}Y_1^dY_2^cY_2^aY_1^b$) are a 0-form on the horizon, and their Hodge dual is a top-form. Any top-form ϕ on a connected manifold can be written as

$$\phi = c\omega + d\psi, \quad (2.7)$$

where ω is the volume form (normalized to unit integral) and c is the integral of ϕ over the manifold. The theorem of Hodge states that for any form ψ , one can always find a form u such that $d*du = d\psi$. In this case, we set $y = -u$. Then the sum of the first four terms on the RHS is

$$c = \int_{M_H} (p_{b;d}\hat{h}^{bd} - \mathcal{R}_{ac}Y_1^aY_2^c + \mathcal{R}_{adcb}Y_1^dY_2^cY_2^aY_1^b) d\hat{S}. \quad (2.8)$$

The first term in the integral does not contribute because it is a divergence.

The Gauss-Godazzi equations, evaluated on the horizon (where the shear and divergence are zero) yield

$$\hat{\mathcal{R}} = \mathcal{R}_{ijkl}\hat{h}^{ik}\hat{h}^{jl} = \mathcal{R} - 2\mathcal{R}_{ijkl}Y_1^iY_2^jY_1^kY_2^l + 4\mathcal{R}_{ij}Y_1^iY_2^j, \quad (2.9)$$

where $\hat{\mathcal{R}}$ is the Ricci scalar associated to the induced metric, and the unhatted quantities are the curvature tensors of the full metric. The integral is therefore equal to

$$\int_{M_H} \left(-\frac{1}{2}\hat{\mathcal{R}} + \frac{1}{2}\mathcal{R} + \mathcal{R}_{ab}Y_1^aY_2^b \right) d\hat{S}. \quad (2.10)$$

From the Einstein equations and the normalization $Y_1 \cdot Y_2 = -1$ we have

$$\frac{1}{2}\mathcal{R} + \mathcal{R}_{ab}Y_1^aY_2^b = 8\pi T_{ab}Y_1^aY_2^b \geq 0, \quad (2.11)$$

where we used the dominant energy condition. Thus we have:

$$\left. \frac{d\hat{\theta}'}{dw'} \right|_{w=0} = \int_{M_H} \left(-\frac{1}{2}\hat{\mathcal{R}} + 8\pi T_{ab}Y_1^aY_2^b \right) d\hat{S} + p'_a p'^a. \quad (2.12)$$

Suppose $\left. \frac{d\hat{\theta}'}{dw'} \right|_{w=0}$ is positive everywhere on the horizon. We then take w' to be a small negative value, thereby looking at a surface slightly outside the horizon on which $\hat{\theta}'$ is now negative. Such a surface is an outer trapped surface, which is forbidden in a stationary regular predictable space-time satisfying the energy conditions. This is because the area of the light-cone of such a surface always shrinks in any time evolution, and hence cannot intersect future null infinity \mathcal{I}^+ (where the area would be infinite). Any region not observable from \mathcal{I}^+ is by definition within the event horizon (and hence not “outer”).

If

$$\int_{M_H} \left(-\frac{1}{2} \hat{\mathcal{R}} + 8\pi T_{ab} Y_1^a Y_2^b \right) d\hat{S}, \tag{2.13}$$

is positive, then it is possible to choose y such that $\left. \frac{d\hat{\theta}'}{dw'} \right|_{w=0}$ is positive everywhere on the horizon, since p' lies on the horizon, and hence is a space-like vector with positive (length)². This leads to an outer trapped surface. Thus, this quantity must be negative or zero. The dominant energy condition $T_{ab} Y_1^a Y_2^b \geq 0$ implies then that

$$\int_{M_H} \hat{\mathcal{R}} d\hat{S} \geq 0. \tag{2.14}$$

If this integral equals zero, then in order to avoid outer trapped surfaces, $T_{ab} Y_1^a Y_2^b = 0$. Thus, the sum of the first four terms on the RHS of (2.6) equals zero, and

$$p'^a{}_{;b} \hat{h}^b{}_a + \mathcal{R}_{abcd} Y_1^a Y_2^b Y_1^c Y_2^d = 0, \tag{2.15}$$

on M_H . Moreover, p'^a must be zero on the horizon since $p'^a p'_a$ is positive-definite. This implies that each term in (2.15) must vanish independently on the horizon. We can then choose the rescaling parameter y such that $\hat{h}^b{}_a p'^a{}_{;b} - \frac{1}{2} \hat{\mathcal{R}} = 0$ on the deformed horizon for small negative w' . This gives rise to a marginally outer trapped surface, which is also forbidden.

Therefore, we arrive at the requirement (2.1), for black hole horizons in asymptotically flat space of any dimensionality. Note, that for the theorem to hold we used the asymptotic flatness, and it does not hold for instance in asymptotically AdS spaces where the dominant energy condition does not hold.

3. The topology of event horizons in higher dimensions

In the previous section we found that for asymptotically flat stationary black holes, in both the four-dimensional case and higher dimensions, the integrated Ricci scalar curvature with respect to the metric induced on the event horizon M_H is positive (2.1). When working in four dimensions, where the event horizon is a two-dimensional manifold, this integral is proportional to the Euler characteristic of the horizon manifold and implies that the topology of the event horizon is that of the two-sphere S^2 .

In addition to (2.1) there is another constraint on the topology of the event horizon as a consequence of the so-called topological censorship [2, 3]. Mathematically it requires that horizon is cobordant to a sphere via a simply connected cobordism. For a d -dimensional

horizon it means that there is a simply connected $(d + 1)$ -dimensional oriented manifold whose boundary is the oriented disjoint union $M_H \cup S^d$. When working in four dimensions, topological censorship also implies that topology of the event horizon is the two-sphere S^2 .

In the following we will use these two conditions and study their implications for the topology of the event horizons in dimensions higher than four.

3.1 Five-dimensional black holes

Consider five-dimensional stationary black holes. Now the horizons are three-manifolds. Thurston introduced eight geometric types in the classification of three-manifolds [6] (see also [7]).² According to this classification there are eight basic homogeneous geometries, up to an equivalence relation, called geometric types. Out of these types one constructs geometric structures, which are spaces that admit a complete locally homogeneous metric.³ Any compact and oriented three-manifold has a decomposition as a connected sum of these basic geometric types.

We consider an orientable, connected, complete and simply connected riemannian three-manifold X which is homogeneous with respect to an orientation preserving group of isometries G . The eight geometric types classify (X, G) . The equivalence relation $(X, G) \sim (X', G')$ holds when there is a diffeomorphism of X onto X' , which takes the action of G onto the action of G' . Out of these types one constructs spaces (geometric structures) $M \simeq X/\Gamma$ where Γ is a discrete subgroup of G . Here the action of Γ on X is free (i. e. the action has no fixed points). M is locally homogeneous with respect to the metric on (X, G) . It is isometric to the quotient of X by Γ .

The first three types in the classification are based on the three constant curvature spaces, the 3-sphere S^3 (Spherical geometry), which has a positive scalar curvature $\mathcal{R} > 0$ and isometry group $G = SO(4)$, the Euclidean space R^3 (Euclidean geometry) with $\mathcal{R} = 0$ and isometry group $G = R^3 \times SO(3)$ and the hyperbolic space H^3 (Hyperbolic geometry) with $\mathcal{R} < 0$ and isometry group $G = PSL(2, C)$. Of these three geometric types, only the S^3 type satisfies the condition (2.1) and is allowed as an horizon.

The next two types are based on $S^2 \times R$ and $H^2 \times R$. Of these two geometric types, only the $S^2 \times R$ type satisfies the condition (2.1) and is allowed as an horizon. In this allowed class we have $S^2 \times S^1$.

The last three geometric types are Nil geometry, Sol geometry and the universal cover of the Lie group $SL(2, R)$.

The Nil geometry: this is the geometry of the three-dimensional Lie group of 3×3 real upper triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

²This classification is called “Thurston Geometrization Conjecture”, and is claimed to have been established by Perelman.

³ M is called locally homogeneous if for any two points x, y in M there are neighborhoods of these two points U_x, U_y and an isometry that maps (x, U_x) to (y, U_y) .

under matrix multiplication (Heisenberg group). We can think about Nil as $(x, y, z) \in R^3$ with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'). \tag{3.2}$$

The Nil metric (the left-invariant metric on R^3) is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2, \tag{3.3}$$

and has $\mathcal{R} = -\frac{1}{2}$. The Nil geometry type does not satisfy the condition (2.1) and is not allowed as an horizon.

The Sol geometry: this is the geometry of the of the Lie group obtained by the semidirect product of R with R^2 . We can think about Sol as $(x, y, z) \in R^3$ with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^z y', z + z'). \tag{3.4}$$

The left-invariant Sol metric is given by

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{3.5}$$

and has $\mathcal{R} = -2$. The Sol geometry type does not satisfy the condition (2.1) and is not allowed as an horizon.

The $\widetilde{Sl(2, R)}$ geometry: this is the geometry of the universal covering of the three-dimensional Lie group of all 2×2 real matrices with determinant one $Sl(2, R)$. The $\widetilde{Sl(2, R)}$ geometry type has $\mathcal{R} < 0$. Thus, it does not satisfy the condition (2.1) and is not allowed as an horizon.

We should note, that for the Nil, Sol and $\widetilde{Sl(2, R)}$ geometries, we considered the natural metrics which lead to locally homogeneous metrics on the event horizons.

Summary: we find that only two geometric types are allowed horizons in five dimensions: the S^3 geometric type and the $S^2 \times R$ geometric type. Indeed, black hole solutions of both geometric types with compact event horizon topologies have been constructed, namely S^3 and $S^2 \times S^1$.

3.2 Six-dimensional black holes

Consider now six-dimensional stationary black holes in asymptotically flat space-times. We will use topological censorship⁴ together with Friedman's classification of four-manifolds and Donaldson's results on smooth four-manifolds, in order to classify possible event horizons.

First we note that oriented cobordism from the event horizon to a four-sphere S^4 exists if and only if the horizon manifold M_H has vanishing Pontrjagin and Steifel-Whitney numbers [9]. In general, two smooth closed n -dimensional manifolds are cobordant iff

⁴In this section we will use topological censorship as oriented cobordism of M_H and S^4 without requiring the five-dimensional manifold to be simply connected.

all their corresponding Steifel-Whitney numbers are equal. If in addition we require the cobordism to be oriented then (when $n = 4k$) their corresponding Pontrjagin numbers are equal. In the following we will study the restriction that these set on the topology of the four-manifold event horizons.

We start by considering simply connected event horizons, that is

$$\Pi_1(M_H) = 0. \tag{3.6}$$

This, in particular, implies that the cohomology groups $H^1(M_H)$ (which is the abelianization of the first homotopy group), and $H^3(M_H)$ (by Hodge duality) vanish. One can use the second cohomology group $H^2(M_H)$ to define an intersection form

$$Q(\alpha, \beta) = \int_{M_H} \alpha \wedge \beta, \tag{3.7}$$

where $\alpha, \beta \in H^2(M_H)$. Q is the basic topological invariant of a compact four-manifold. Note that, since the four-sphere S^4 has zero second cohomology group, all its intersection numbers vanish and

$$Q(S^4) = (0). \tag{3.8}$$

Q is symmetric, non-degenerate with $rank(Q) = b_2 \equiv dim H^2(M_H)$, and can be diagonalized over R . The signature σ of a four-manifold is defined by the difference of positive and negative eigenvalues of Q . It can be expressed using the Hirzebruch signature theorem as

$$\sigma(M_H) = \frac{1}{3} \int_{M_H} p_1, \tag{3.9}$$

where p_1 is the first Pontrjagin class, which can be expressed using the Riemann curvature as

$$p_1 = -\frac{1}{8\pi^2} Tr R \wedge R. \tag{3.10}$$

Since $p_1(S^4)$ vanishes, topological censorship implies then that the signature of $Q(M_H)$ vanishes

$$\sigma(Q(M_H)) = 0. \tag{3.11}$$

Mathematically, the signature is cobordant invariant.

Consider next the Stiefel-Whitney classes

$$\omega_i \in H^i(M_H, Z_2). \tag{3.12}$$

For a compact, simply-connected orientable manifold $\omega_1 = \omega_3 = 0$. ω_2 is the obstruction to a spin-structure. Although $\omega_2(S^4) = 0$, oriented cobordism does not imply that the second Stiefel-Whitney class of M_H is zero. In other words, M_H is not necessarily a spin manifold.

The intersection form Q is actually defined on the lattice $H^2(M_H, Z)$ and is a unimodular ($det(Q) = \pm 1$) symmetric bilinear form over the integers. One says that Q is of even type if

$$Q(\alpha, \alpha) \in 2Z \tag{3.13}$$

for all $\alpha \in H^2(M_H, Z)$. If $\omega_2 = 0$ then Q is even (as implied by Wu's formula [9]). Thus, event horizons which are spin manifolds are characterized as topological four-manifolds by an intersection form $Q(M_H)$, which has vanishing signature and is of even type. When the event horizons are not spin manifolds, $\omega_2 \neq 0$ and $Q(M_H)$ is odd. In this case there are two topological four-manifolds M_H for a given intersection form. They are distinguished by the Kirby-Siebenmann invariant, which is zero if $M_H \times S^1$ is smooth and one if $M_H \times S^1$ is not smooth.

In the following we will use combined results of Friedman's classification and Donaldson's theorems (see [10, 11]). In the classification of possible intersection forms of M_H we distinguish two cases:

- (i) $Q(M_H)$ is positive definite,
- (ii) $Q(M_H)$ is indefinite.

Consider first the case when $Q(M_H)$ is positive definite. If $Q(M_H)$ is even then M_H is homeomorphic to the four-sphere S^4 (see [10, corollary (2.27)]). If $Q(M_H)$ is odd then M_H is homeomorphic to a connected sum of CP^2 's. However, since $Q(CP^2) = (1)$, the signature of the connected sum is nonzero, and this is not an allowed horizon.

If $Q(M_H)$ is indefinite, then if it is even it can be written as (Hasse and Minkowski)

$$Q(M_H) = aE_8 + bH, \quad a, b \in Z \quad b \neq 0, \tag{3.14}$$

where E_8 is the Cartan matrix of the Lie algebra E_8 and

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.15}$$

is the intersection form of $S^2 \times S^2$. Since the signature $\sigma(E_8) = 8$ and we require that $\sigma(Q(M_H)) = 0$ this implies that $a = 0$. The basic case is $b = 1$ and M_H is $S^2 \times S^2$. If we take $b > 1$ we will get a connected sum of $S^2 \times S^2$.

We should note, however, that by using connected sums there is a way to construct other event horizons whose intersection form has vanishing signature and is of even type. Consider for instance a $K3$. Its intersection form is $Q(K3) = -2E_8 + 3H$. Since its signature is nonzero, $K3$ is not an allowed horizon. However, if we take a connected sum of the $K3$ and $-K3$, where $-K3$ has an opposite orientation we get an even intersection form with vanishing signature, since $Q(-K3) = -Q(K3)$.

If $Q(M_H)$ is indefinite and odd then M_H is a connected sum of $\pm CP^2$'s, where $-CP^2$ has the opposite orientation of CP^2 and $Q(-CP^2) = -Q(CP^2) = (-1)$.

In the following we consider the possible event horizons up to the connected sum operation.

Summary. We found that if the horizon is simply connected then it is homeomorphic to S^4 or to $S^2 \times S^2$, up to connected sum. Note that both S^4 and $S^2 \times S^2$ are spin manifolds.

Consider next the case when M_H is not simply connected. When $Q(M_H)$ is positive definite, one can relax the condition that $\Pi_1(M_H) = 0$ by requiring only that there are no non-trivial homomorphisms of $\Pi_1(M_H)$ into $SU(2)$. This implies that every flat

$SU(2)$ bundle over M_H is trivial and that $H_1(M_H)$ being the abelianization of $\Pi_1(M_H)$ vanishes [10]. This allows $\Pi_1(M_H)$ to be any finite simple nonabelian group. With this relaxed condition we get four-manifold event horizons, whose universal cover is S^4 .

There are three other non-simply connected cases that we would like to explore. First, consider T^4 . Its intersection form is $Q(T^4) = 3H$ and it is not excluded by the previous discussion from being an event horizon. However, it does not satisfy our curvature condition (2.1). Next consider $S^3 \times S^1$. It is not ruled out by our analysis since it has vanishing Pontrjagin and Steifel-Whitney numbers. Also, it satisfies (2.1). The last examples are $\Sigma_g \times \Sigma_h$, where Σ_g and Σ_h are Riemann surfaces of genus g and h respectively, and we have assumed that the induced metric decomposes as a direct (unwarped) sum. Assuming a product metric, the condition (2.1) reads

$$(g - 1)Vol(\Sigma_h) + (h - 1)Vol(\Sigma_g) < 0. \tag{3.16}$$

This can be satisfied by $\Sigma_h = S^2$ and

$$g < 1 + \frac{Vol(\Sigma_g)}{Vol(S^2)}. \tag{3.17}$$

We cannot exclude, a priori, the possibility that the ratio of volumes in (3.17) can be as large as we want and therefore all genera g are allowed. We encountered above the case $g = 0$, namely the horizon $S^2 \times S^2$. The intersection form $Q(S^2 \times \Sigma_g) = H$ and it is not ruled out by topological censorship.

Summary. We found that if the event horizon has vanishing first homotopy group then it is homeomorphic to S^4 or $S^2 \times S^2$. If the event horizon has finite simple nonabelian first homotopy group and positive intersection form, then its universal cover is homeomorphic to S^4 . We found other allowed non-simply connected cases $S^3 \times S^1$ with first homotopy group Z and $S^2 \times \Sigma_g$ with first homotopy group⁵ $\Pi_1(\Sigma_g)$.

3.3 Comments on higher dimensions

In the following we will make some comments on the classification of the event horizons above six dimensions. The event horizons M_H are now closed differentiable n -manifolds with dimension n higher than four, cobordant to the n -sphere S^n . If M_H is homotopic to S^n , then by the generalized Poincare conjecture (proven when $n > 4$) M_H is homeomorphic to S^n [12].

An important concept in differential topology is that of h-cobordism. Two cobordant n -manifolds are h-cobordant if their inclusion map in the $n + 1$ -dimensional manifold are homotopy equivalent. The h-cobordism theorem [12] implies that if the horizon manifold M_H is h-cobordant to S^n then it is diffeomorphic to S^n . Note, however, that h-cobordism is a stronger requirement than what is implied by topological censorship.

Let us introduce the concept of spin cobordism. Mathematically, it requires in our context the existence of an $(n + 1)$ -dimensional compact spin manifold, whose boundary

⁵ $\Pi_1(\Sigma_g)$ is generated by $a_1, b_1, \dots, a_g, b_g$ with the relation $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$.

is the oriented disjoint union of S^n and M_H . In particular, M_H is a spin manifold whose spin structure is induced from that of the $(n + 1)$ -dimensional manifold. The concept of spin cobordism may be relevant, since we are mainly interested in higher-dimensional black holes solutions to supergravity equations as the low energy effective description of the superstring equations. Therefore, we would like the geometry to accommodate fermions. Note, that in the classification of the previous section, spin cobordism would have implied that the intersection form is of even type.

There are several useful results that help in the classification of the event horizons of dimensions higher than four:

- $n = 5$ For five-dimensional manifolds with vanishing second Steifel-Whitney class , there exists a classification of all possible closed simply connected manifolds [12]. The manifolds are in 1-1 correspondence with finitely generated abelian groups.
- $n = 6$ Six-dimensional closed manifolds with vanishing first and second homotopy groups $\Pi_1 = 0$ and $\Pi_2 = 0$ (2-connected) are homeomorphic to S^6 or connected sum of copies of $S^3 \times S^3$ [12].
- $n = 2k$ There are general results which enumerate the $(k - 1)$ -connected $2k$ -manifolds (Wall) [12].
- $n \geq 5$: If M_H is n -dimensional compact, simply connected and spin cobordant to S^n , it is obtained from S^n by doing surgery on spheres of codimension greater than two [13].

We will leave the complete analysis of the possible topologies of event horizons of stationary black holes in asymptotically flat space-times with $dim M_H > 4$ to the future.

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